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SELECTED PROBLEMS OF NUMERICAL CALCULATIONS OF DIFFER-INTEGRALS OF NON-INTEGER ORDERS

ABSTRACT *This paper presents methods of calculating numerically differ-integrals of non-integer orders. We evaluate the Riemann-Liouville formula in the context of the accuracy of the calculations. The point of reference is another popular formula – Grünwald-Letnikov – often used in technical applications because of its simplicity. By implementing transformations to the core integrand of the Riemann-Liouville formula we want to present the possible ways of reducing absolute errors while using this formula. We also test accuracy abilities of some methods of numerical integration: three Gauss quadratures and one Newton-Cotes formula. The test bed will be an interesting exponential function – often used in technical, practical applications. We will not discuss complexity of numerical calculations in details. We will focus solely on minimization of the absolute errors.*

Keywords: *numerical calculations, differential integrals, accuracy, errors, exponential function, fractional orders*

1. INTRODUCTION

Fractional calculus plays recently an important role in many scientific areas. Fractional Order Derivatives and Integrals (FOD/FOI) are a natural extension of the terms: “integral” and “derivative” as we know them. This extension enables

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better identification and analysis of physical phenomena and more precise control of physical processes and devices.

The very new and interesting field, where the FOD/FOI could be applied is the modelling of real dynamical systems, phenomena and dynamical system control using PID controllers (PID stands for Proportional Integral Derivative). Replacement of a conventional PID controller with fractional one to approximate fractional-order "reality" rewards always with achieving faster steady state.

But there are still problems with accuracy of numerical calculations of FOD/FOI [9, 10, 11].

In this paper are presented two methods of calculating FOD/FOI and compared due to their accuracy: the Riemann-Liouville fractional derivative and integral and Grünwald-Letnikov fractional differ-integral, but only the Riemann--Liouville formula will be extensively discussed. The Grünwald-Letnikov formula will be used for comparing purposes only in the context of calculation accuracy.

The paper is organised as follows: Firstly, basic definitions of FOD and FOI are given. A short review of numerical methods used in the process of calculating FOD/FOI by Riemann-Liouville formula is presented – section 3. Special transformations to the core integrand of Riemann-Liouville formula are elaborated – section 4. After that we list all applied methods in details as well as the goals and the tested functions – section 5. Section 6 – the main results are presented: the accuracy comparison between the formulas of calculating FOD/FOI, applied methods of numerical integration, the "pure" and transformed Riemann-Liouville formula. Finally, the conclusions are drawn – section 7.

2. MATHEMATICAL PRELIMINARIES

There are several formulas to calculate FOD/FOI numerically. One of them is the Grünwald-Letnikov one and Riemann-Liouville is another one, formulas [1, 3, 10, 13].

2.1. Grünwald-Letnikov formula of differ-integral of a fractional-order (GrLET)

The derivative of a real order $\nu > 0$ (for the integral we use order $-\nu < 0$) of a continuous bounded function $f(t)$ is defined as follows:

$$ {}_{t_0}D_t^{\nu} f(t) = \lim_{\substack{h\to 0 \\ t-t_0=kh}} \frac{\sum_{i=0}^{\frac{t-t_0}{h}} a_i^{(\nu)} f(t-hi)}{h^{\nu}} \tag{1} $$

where

$$ a_i^{(\nu)} = \begin{cases} 1 & \text{for} \quad i=0 \\ a_{i-1}^{(\nu)}\left(1-\frac{1+\nu}{i}\right) & \text{for} \quad i=1,2,3,\ldots \end{cases} \tag{2} $$

GrLET formula is a pretty straightforward formula – often used in practical applications due to its simplicity. The formula is derived from a differential quotient. Its accuracy depends strongly on the amount of the coefficients a_i and accuracy of their calculations. In practical applications, in control systems we use 500-600 of them. This assures the accuracy of $10e-04$. We want to challenge this level of accuracy with the next method.

2.1. Riemann-Liouville formulas of fractional-order derivative and integral (RL)

Definite Riemann-Liouville integral of the real function $f(t)$ of the $\nu > 0$ order is defined as follows:

$$ {}_{t_0}I_t^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{t_0}^{t} (t-\tau)^{\nu-1} f(\tau) d\tau \tag{3} $$

where: t_0, t – integration range, which complies with the condition $-\infty < t_0 < t < \infty$,
$\Gamma(\nu)$ – Euler's Gamma function.

Before we define the Riemann-Liouville derivative, we have to describe the natural number n, which complies with the condition:

$$ n = [\nu] + 1 \tag{4} $$

The Riemann-Liouville derivative of the real function $f(t)$ of the $\nu > 0$ order is defined as follows:

$$ {}_{t_0}D_t^{\nu} f(t) = \sum_{i=0}^{n-1} \frac{(t-t_0)^{i-\nu} f^{(i)}(t_0)}{\Gamma(i+1-\nu)} + \frac{1}{\Gamma(n-\nu)} \int_{t_0}^{t} (t-\tau)^{n-\nu-1} f^{(n)}(\tau) d\tau \tag{5} $$

RL formulas are derived from multiple integrals. This property determines the way they are calculated – it involves numerical methods of integration, which implies that the accuracy of calculations depends not only on the shape and character of the integrand, the amount of the sample points but also on the abilities of a particular method of numerical integration as well.

3. SHORT REVIEW OF FUNDAMENTAL METHODS OF NUMERICAL INTEGRATION

In the process of calculating differ-integrals it is necessary to calculate a value of the definite integral over the range $[t_0, t]$. Usually it is approximated with the following formula

$$\int_{t_0}^{t} f(t)dt = \sum_{k=0}^{L} A_k f(t_k) + R. \tag{6}$$

Right side of the equation is called quadrature, in which t_k – denotes quadrature nodes, A_k – quadrature coefficients (weights), L – number of intervals in interpolation and R – remainder.

The above formula is shared by all quadratures. The difference lies in the algorithms of calculating their nodes and coefficients.

The interpretation of quadrature coefficients A_k (so called weights) is the width of the subintervals the range of integration is divided into.

- In case of Newton-Cotes Quadrature (Rectangle, Trapezoidal and Simpson Rule) the integration range is divided into subintervals, which are of equal width.
- In Newton Cotes' Midpoint Rule the sample point is taken from the middle of the subinterval.
- Nodes of Gauss quadratures are determined by abscissas of the applied polynomials in approximation.

Newton-Cotes quadratures may be applied to almost all types of integrands. The trapezoidal and Simpson Rules are the most often used.

They are called "closed formulas", which implies that the values of endpoints of the integration range must be determinable. Therefore they cannot be applied to integrands which have singularities, asymptotes, etc. [2, 8]

Gauss quadratures, other than the Gauss-Legendre quadrature (weight function equals (1) may be applied only to selected kinds of integrands. This is due to relationship with their weight functions [4, 6, 7].

TABLE 1
Important parameters used in integration rules

Method/ weight function	h / A_k	t_k	$R \le$
Newton-Cotes Midpoint Rule	$h = \frac{t - t_0}{L}$	$t_k = t_0 + (k + 1/2)h$	$\frac{h^3}{24}\left\|f^{(II)}(\zeta)\right\|,\ \zeta \in [t_0, t]$
Gauss-Legendre Quadrature $p(x) = 1$	$A_k = \frac{2}{(1 - t_k^2)[P_n'(t_k)]^2}$	Abscissas of the Legendre polynomial $P_n(x)$ of the desired grade x_k. $t_k = \frac{t - t_0}{2} x_k + \frac{t - t_0}{2}$	$\frac{t - t_0}{2016000} f^{(VI)}(\zeta),\ \zeta \in [t_0, t]$
Gauss-Laguerre Quadrature $p(x) = e^{-x}$	$A_k = \frac{(n!)^2}{x_k [L_n'(x_k)]^2}$	Abscissas of the Laguerre polynomial $L_n(x)$ of the desired grade x_k.	$\frac{(n!)^2}{(2n)!} f^{(2n)}(\zeta),\ \zeta \in \langle 0; +\infty)$

A special modification of Gauss quadrature named Gauss-Kronrod quadrature was used in form of tabulated values of nodes and weights. This method is based on Gauss-Legendre quadrature. There is no need to overlap all the details about it. It is enough to say that the G7/K15 the so called Gauss-Kronrod Pair (we apply it along with the G30/K61 one) includes nodes of the 7-point Gauss-Legendre Quadrature + 8 new ones and all 15 new coefficients [5].

4. TRANSFORMATIONS AND SPECIAL SUBSTITUTE EXPRESSIONS EXPLAINED

The integrand of RL formulas (3, 5) is not only fast-changing but also includes a singularity at the end of the integration range (Fig. 1).

That means, that knowing the weaknesses of numerical methods of integration, our goals were to transform the integrand of the RL formulas (3, 5) to:

- remove the singularity,
- make it "smoother".

Two transformations and special substitute expressions were used to fulfil both tasks. The transformations idea is similar to substituting to evaluate the integrals analytically. Instead of simplifying the complicated integrand expression we targeted to obtain desired shape and properties of the transformed integrand.

The IMT Transformation (m1 RL) proposed in 1970 by three Japanese mathematicians: Iri, Moriguti and Takasawa is based on the idea of transforming the independent variable to make all derivatives of the transformed integrand at the ends of the integration range. The substitute expression: $t-\tau \rightarrow e^{1-1/t}$ applied to formulas (3, 5) made it happen.

The second transformation – we call it "the inverse transformation" (m2 RL) is inspired by the IMT Transformation. The variable changes: $t-\tau \rightarrow 1/u^{\alpha}$, $\alpha = R_{+}$ and $u-1=w$ applied to formulas (3, 5) convert the improper integral into one, so after extracting the weight function $p(x)=e^{-x}$ it can be then calculated with higher accuracy by the Gauss-Laguerre quadrature formula, which was developed to deal with such problems. Additionally – with the parameter α we can increase this accuracy. As is noticeably, there exists a very close accuracy relation between the order of the calculated differ-integral and the value of the parameter α. We will use this property to our advantage.

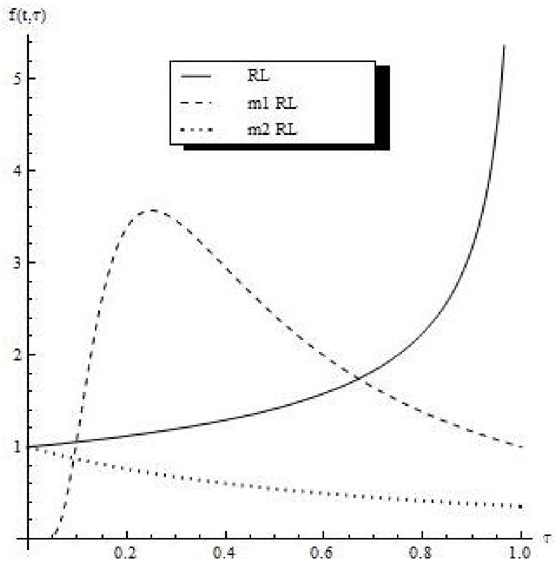

Fig. 1. Graph of the original and the transformed RL formulas

Both transformations can be applied to each function under the condition of transforming the calculated expression and possibly the integration range via the substituting expression.

Our goal was also to compare both transformations against the accuracy of calculations. And how they compete against the Grünwald-Letnikov method.

5. LIST OF THE APPLIED METHODS AND TECHNIQUES. MAIN GOALS AND THE TESTED FUNCTION

We used following formulas to calculate differ-integrals:

- Riemann-Liouville fractional order integral and derivative (RL),
- Modified Riemann-Liouville differ-integral via the IMT Transformation (m1 RL) and the inverse transformation (m2 RL).

Additionally we used Grünwald-Letnikov differ-integral formula (GrLET) as a reference method – for comparing purposes in the context of accuracy calculations.

Our C++ programs developed especially for the purpose of this experiment used the following methods of numerical integration while applying formulas (RL, m1RL, m2RL):

- Newton-Cotes quadrature, Midpoint Rule (NCM),
- Gauss-Legendre quadrature (GaLEG),
- Gauss-Laguerre quadrature (GaLAG),
- Gauss-Kronrod quadrature (GaKRO).

The goal was to obtain the results charged with the possible smallest absolute error using the possibly smallest, arbitrally chosen, number of sampling points – possibly less than 600:

- For the method GrLET and NCM we use L=4, 15, 32, 61 and 600 intervals.
- For GaLEG and GaLAG – L=4, 15, 32 and sometimes (where emphasized) more, but not more than 42 intervals.

It is widely known, that a number of L greater that 30-40 for the GaLEG and GaLAG methods often causes the error rise rapidly. Sometimes 100% and more! That is why there are empty fields in all tables with results for these methods.

- For GaKRO we use the following pairs: G7/K15 and G30/K61 which are the industry standard in numerical integration. They correspond to L=15 and 61 intervals of the integration respectively.

We tested the function which is very often used in technical applications, especially in control systems. It also states the typical member in solutions of linear, stationary differential equations of the integer orders:

$$f(t) = e^{-2t},\ t \in (0,1) \tag{7}$$

This function is able to highlight pros and cons of every applied method of numerical integration.

We calculated two kinds of expressions:

- the integral ${}_{t_0}I_t^{\nu} f(t),\ \nu = 0.2, 0.5, 0.8$;
- and the derivative ${}_{t_0}D_t^{\nu} f(t),\ \nu = 0.2, 0.5, 0.8$.

As presented, the order of the calculated expression is closely linked to the shape of the integrand in the RL, m1 RL and m2 RL. This will also test various abilities of applied methods of numerical methods of integration.

6. THE RESULTS

Firstly, the integral ${}_0I_1^{\nu} f(t), \nu = 0.2, 0.5, 0.8$ of the function (7) were calculated. The modified RL formula via m1 RL assumes the form (8) and m2 RL (9). The results are presented in Tables 2a-2c and in Table 2d the optimal values of α for the method m2 RL for every ν (+ denotes higher values of the absolute error).

$$ {}_0I_1^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^1 \frac{\left(e^{1-1/t}\right)^{\nu} e^{-2\left(1-e^{1-1/t}\right)}}{t^2} dt \tag{8} $$

$$ {}_0I_1^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^{+\infty} e^t \frac{\alpha\, e^{-2\left(1-1/(1+t)^{\alpha}\right)}}{\left(1/(1+t)^{\alpha}\right)^{\nu-1}(1+t)^{\alpha+1}} dt \tag{9} $$

TABLE 2a
Obtained values of the absolute error for $\nu = 0.2$

L	GrLET	RL GaLEG	RL NCM	RL GaKRO	m1 RL GaLEG	m1 RL NCM	m1 RL GaKRO	m2 RL GaLAG
4	3.207 e-02	6.194 e-02	7.356 e-02	-	1.019 e-02	3.207 e-02	-	6.450 e-02
15	8.190 e-03	3.786 e-02	5.639 e-02	3.493 e-02	2.711 e-04	2.676 e-03	2.829 e-06	9.530 e-03
32	4.209 e-03	**2.816 e-02**	4.844 e-02	-	1.297 e-06	1.455 e-04	-	**1.225 e-03**
61	4.187 e-03	-	2.888 e-02	**1.991 e-02**	**9.380 e-08** L=34	7.297 e-06	**4.922 e-10**	-
600	**2.259 e-04**	-	**2.697 e-02**	-	-	**5.042 e-09**	-	-

TABLE 2b
Obtained values of the absolute error for $\nu = 0.5$

L	GrLET	RL GaLEG	RL NCM	RL GaKRO	m1 RL GaLEG	m1 RL NCM	m1 RL GaKRO	m2 RL GaLAG
4	7.388 e-02	1.462 e-02	2.040 e-02	-	8.253 e-03	1.884 e-03	-	8.020 e-02
15	4.379 e-02	8.830 e-03	1.765 e-02	3.486 e-03	4.285 e-03	1.676 e-03	2.062 e-05	8.410 e-03
32	9.946 e-03	2.045 e-03	8.147 e-03	-	**1.161 e-08**	1.213 e-05	-	**2.124 e-04**
61	5.243 e-03	**1.509 e-03** L=42	1.509 e-03	**8.542 e-04**	-	1.175 e-06	**7.029 e-13**	-
600	**5.357 e-04**	-	**1.885 e-03**	-	-	**3.265 e-08**	-	-

TABLE 2c
Obtained values of the absolute error for $\nu = 0.8$

L	GrLET	RL GaLEG	RL NCM	RL GaKRO	m1 RL GaLEG	m1 RL NCM	m1 RL GaKRO	m2 RL GaLAG
4	9.642 e-02	1.868 e-03	7.170 e-03	-	1.535 e-03	5.739 e-03	-	8.077 e-02
15	5.737 e-02	2.674 e-04	1.647 e-03	1.900 e-04	3.198 e-06	1.399 e-04	3.397 e-06	2.641 e-03
32	1.308 e-02	**8.189 e-05**	8.321 e-04	-	**2.795 e-09**	2.798 e-05	-	**6.850 e-05**
61	6.898 e-03	-	4.831 e-04	**2.001 e-05**	-	7.695 e-06	**6.217 e-15**	-
600	**7.051 e-04**	-	**7.606 e-05**	-	-	**7.953 e-08**	-	-

TABLE 2d
Optimal values of α for the method m2 RL for every ν calculated

α	$\nu = 0.2$	$\nu = 0.5$	$\nu = 0.8$
5.792	1.225 e-03	+	+
3.050	+	2.124 e-04	+
2.200	+	+	6.829 e-05

Next, the derivative ${}_0D_1^{\nu} f(t), \nu = 0.2, 0.5, 0.8$ of the function (7) were calculated. If we assume that t_0 equals zero and (4) – the modified RL formula via m1 RL takes the form (10) and m2 RL (11). The results are presented in Tables 3a-3c. The optimal values of α for the method m2 RL for every ν are shown in Table 3d (+ denotes the higher value of the absolute error).

$$ {}_0D_1^{\nu} f(t) = \sum_{i=0}^{n-1} \frac{1^{i-\nu} e^{-2t}(0)}{\Gamma(i+1-\nu)} + \frac{1}{\Gamma(\nu)} \int_0^1 \frac{-2e^{-2\left(1-e^{1-1/t}\right)} \left(e^{1-1/t}\right)^{\nu}}{t^2} dt \tag{10} $$

$$ {}_0D_1^{\nu} f(t) = \sum_{i=0}^{n-1} \frac{1^{i-\nu} e^{-2t}(0)}{\Gamma(i+1-\nu)} + \frac{1}{\Gamma(\nu)} \int_0^{+\infty} e^t \frac{-2\alpha\, e^{-2\left(1-1/(1+t)^{\alpha}\right)}}{\left(1/(1+t)^{\alpha}\right)^{\nu-1} (1+t)^{\alpha+1}} dt \tag{11} $$

TABLE 3a
Obtained values of the absolute error for $\nu = 0.2$

L	GrLET	RL GaLEG	RL NCM	RL GaKRO	m1 RL GaLEG	m1 RL NCM	m1 RL GaKRO	m2 RL GaLAG
4	2.706 e-02	3.792 e-03	1.434 e-02	-	3.071 e-03	1.148 e-02	-	1.605 e-01
15	7.109 e-03	5.384 e-04	3.294 e-03	3.779 e-04	6.397 e-06	2.798 e-04	6.974 e-06	5.215 e-03
32	3.324 e-03	1.638 e-04	1.664 e-03	-	**5.429 e-09**	5.596 e-05	-	**1.366 e-04**
61	1.742 e-03	**6.990 e-05** L=42	9.663 e-04	**4.001 e-05**	-	1.537 e-05	**1.182 e-14**	-
600	**1.762 e-04**	-	**1.521 e-04**	-	-	**1.591 e-07**	-	-

TABLE 3b
Obtained values of the absolute error for $\nu = 0.5$

L	GrLET	RL GaLEG	RL NCM	RL GaKRO	m1 RL GaLEG	m1 RL NCM	m1 RL GaKRO	m2 RL GaLAG
4	3.771 e-02	2.923 e-02	4.810 e-02	-	1.651 e-02	3.768 e-03	-	1.606 e-01
15	9.119 e-03	8.570 e-03	2.384 e-02	6.970 e-03	3.128 e-05	1.892 e-04	4.124 e-05	2.590 e-03
32	4.203 e-03	**4.090 e-03**	1.629 e-02	-	**2.323 e-08**	2.427 e-05	-	**4.249 e-04**
61	2.190 e-03	-	1.181 e-02	**1.708 e-03**	-	6.338 e-06	**1.423 e-12**	-
600	**2.212 e-04**	-	**3.770 e-03**	-	-	**6.530 e-08**	-	-

TABLE 3c
Obtained values of the absolute error for $\nu = 0.8$

L	GrLET	RL GaLEG	RL NCM	RL GaKRO	m1 RL GaLEG	m1 RL NCM	m1 RL GaKRO	m2 RL GaLAG
4	1.203 e-02	1.239 e-01	1.471 e-01	-	2.032 e-02	6.414 e-02	-	1.298 e-01
15	3.698 e-03	7.571 e-02	1.127 e-01	6.987 e-02	5.427 e-04	5.352 e-03	5.659 e-06	1.937 e-02
32	1.756 e-03	5.631 e-02	9.688 e-02	-	2.594 e-06	2.911 e-04	-	**2.449 e-03**
61	9.252 e-04	**5.563 e-02** L=33	8.518 e-02	**3.981 e-02**	**1.876 e-07** L=34	1.459 e-05	**9.844 e-10**	-
600	**9.443 e-05**	-	**5.394 e-02**	-	-	**1.008 e-08**	-	-

TABLE 3d
Optimal values of α for the method m2 RL for every ν calculated

α	$\nu = 0.2$	$\nu = 0.5$	$\nu = 0.8$
5.86	+	+	2.449 e-03
3.05	+	4.249 e-04	+
2.20	1.366 e-04	+	+

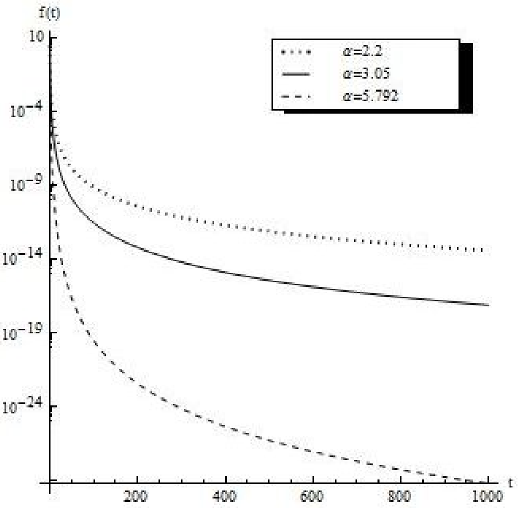

Fig. 2. The influence of the parameter α on the shape of the integrand (m2 RL)

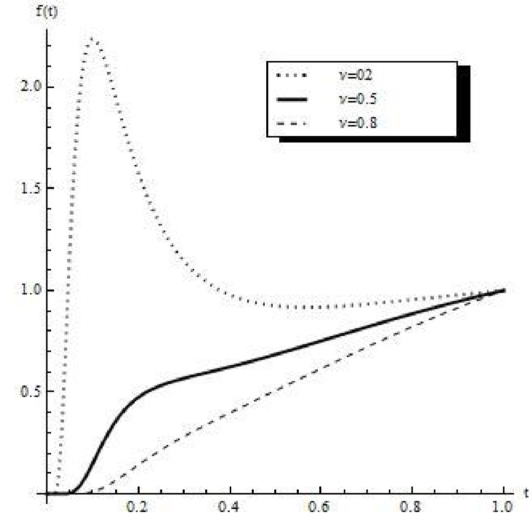

Fig. 3. The influence of the order of the calculated expression ν on the shape of the integrand (m1 RL)

7. CONCLUSIONS

The results presented in section 6 as well as the results obtained in previous experiments of this kind with other types of functions [14] allow us to formulate the following conclusions.

The shape and the characteristics of the integrand does not influence calculations accuracy when applying the Grünwald-Letnikov formula, but the number of the coefficients and how accurately they are calculated – does. Using 600 of them we obtain the results with the average accuracy up to $10e-04$.

The shape and the characteristics of the integrand do influence accuracy of the calculations when using the "pure" and transformed form of Riemann-Liouville formula.

The Gauss-Legendre and Gauss-Kronrod quadratures applied to *unmodified* Riemann-Liouville formula can compete against Grünwald-Letnikov method in the context of accuracy calculations. In some cases the errors obtained by these methods with only a handful of sample points (L=32 against L=600) are one order lower than in the reference method.

The cases in which Gauss methods loose in accuracy against Grünwald-Letnikov are understandable because the characteristics of the tested function incorporated into Riemann-Liouville formulas is fast-changing and has the singularity at the endpoint of the integration range.

The Newton-Cotes' Midpoint Rule is a universal tool. Not only it does not depend so strongly as the Gauss quadratures on the shape and changeability of the integrand. Additionally – it can be applied (similarly to the applied Gauss Rules) to integrands which have singularities at the endpoints of the integration range.

Applying both proposed transformations to the core integrand of the Riemann-Liouville formula prior to the calculations, using methods of numerical integration, increases the accuracy of calculations in a noticeably way with drastically reduced amount of sample points. This again reduces complexity of the calculations (in the meaning of reducing the amount of sample points).

After applying the IMT Transformation to the Riemann-Liouville formulas the accuracy was raised 5-11 times. In case of the inverse transformation and application of Gauss-Laguerre quadrature formula – it increased by 1-2 times as compared with the reference method – Grünwald-Letnikov formula (Tab. 2a-2c and Tab. 3a-3c).

The advantage of the application of the transformed Riemann-Liouville formulas can be the drastically reduced amount of sample points – Gauss methods used maximum 32-61 sample points – 5-10% of the sample points used by the reference method (notice Tab. 2a-2c and Tab. 3a-3c). The reference method needed about 70.000.000 of them to reach the same level of accuracy.

The manipulations of the parameter α in the inverse transformation allow to speed up the convergence of the integrand and increase the accuracy of the calculations. The values of this parameter should be reduced proportionally to the fractional order of the integral calculated and increased proportionally to the fractional order of the derivative calculated (see Tab. 2d and Tab. 3d). The value of α cannot be optimally adjusted automatically as it depends on the tested function (see below).

As it can be noticed the shape of the integrand changes as the order of the calculated expression rises or lowers. This influences the accuracy of the calculations. The manipulation of the α parameter allows to compensate for these harmful changes and keep the level of the absolute error unchanged low.

The logic of the algorithms of our tools – with the exception for the Gauss-Kronrod program – needed only the degree of the desired polynomial as an input data. All other constructions were built "on the fly" (the polynomial itself, its derivative, abscissas and weights). In practical applications we can and should use tabulated values of abscissas and weights which were the subject of standardization by the U.S. Department of Commerce years ago. This can still reduce the complexity of calculations which then can make the methods presented perfectly suitable in practical applications.

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WYBRANE PROBLEMY NUMERYCZNYCH OBLICZEŃ CAŁEK RÓŻNICZKOWYCH RZĘDÓW UŁAMKOWYCH

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STRESZCZENIE *Niniejszy artykuł prezentuje metody numerycznego obliczania pochodnych i całek niecałkowitych rzędów. Staramy się oszacować poziom dokładności obliczeń za pomocą wzoru Riemanna-Liouville'a. Punktem odniesienia będzie inna znana metoda*

Grünwalda-Letnikova – bardzo popularna w zastosowaniach technicznych ze względu na swą prostotę. Zastosowane przekształcenia funkcji podcałkowej we wzorze Riemanna-Liouville'a są naszą propozycją redukcji wartości błędów bezwzględnych w zastosowaniach tej metody. Sprawdziliśmy także, jaką dokładność oferują zastosowane metody całkowania numerycznego: trzy kwadratury Gaussa oraz kwadratura Newtona-Cotesa. Kryterium stanowiła dokładność obliczeń pochodnych i całek niecałkowitych rzędów funkcji wykładniczej – często używanej w praktycznych, technicznych zastosowaniach. W niniejszym artykule nie podejmujemy tematu złożoności obliczeniowej, skupiamy się jedynie na zmniejszaniu wartości błędów bezwzględnych.